Time-correlated model error in (ensemble) Kalman smoothers

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Outline

1. Introduction and motivation
2. Error in model error
3. Weak-constraint solution to Kalman Smoother
4. Uni-dimensional example
5. Introducing errors
6. Small error approximation
7. Conclusions, future work

1. Setup: smoothing problem

$\mathbf{x}^t \in \mathcal{R}^{N_x}$ **Model variables**

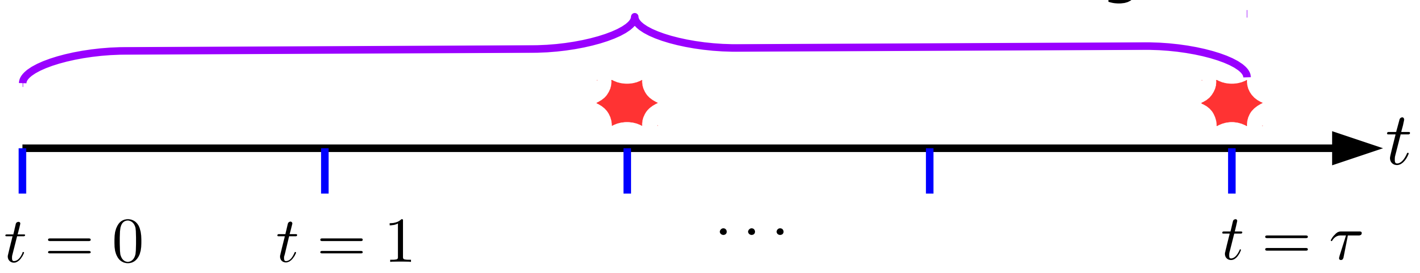
$\mathbf{y}^l \in \mathcal{R}^{N_y}$ **Observations**

$$\mathbf{x}^t = m^{(t-1)\to t}\left(\mathbf{x}^{t-1}\right) + \mathbf{v}^t$$

$$\mathbf{y}^l = h^l\left(\mathbf{x}^{t=l}\right) + \boldsymbol{\eta}^l$$

$$\{\mathbf{x}^0, \mathbf{v}^t, \boldsymbol{\eta}^l\}\ r.v.,\ \ \mathbf{x}^0 \perp \mathbf{v}^t \perp \boldsymbol{\eta}^l$$

assimilation window: **smoothing**



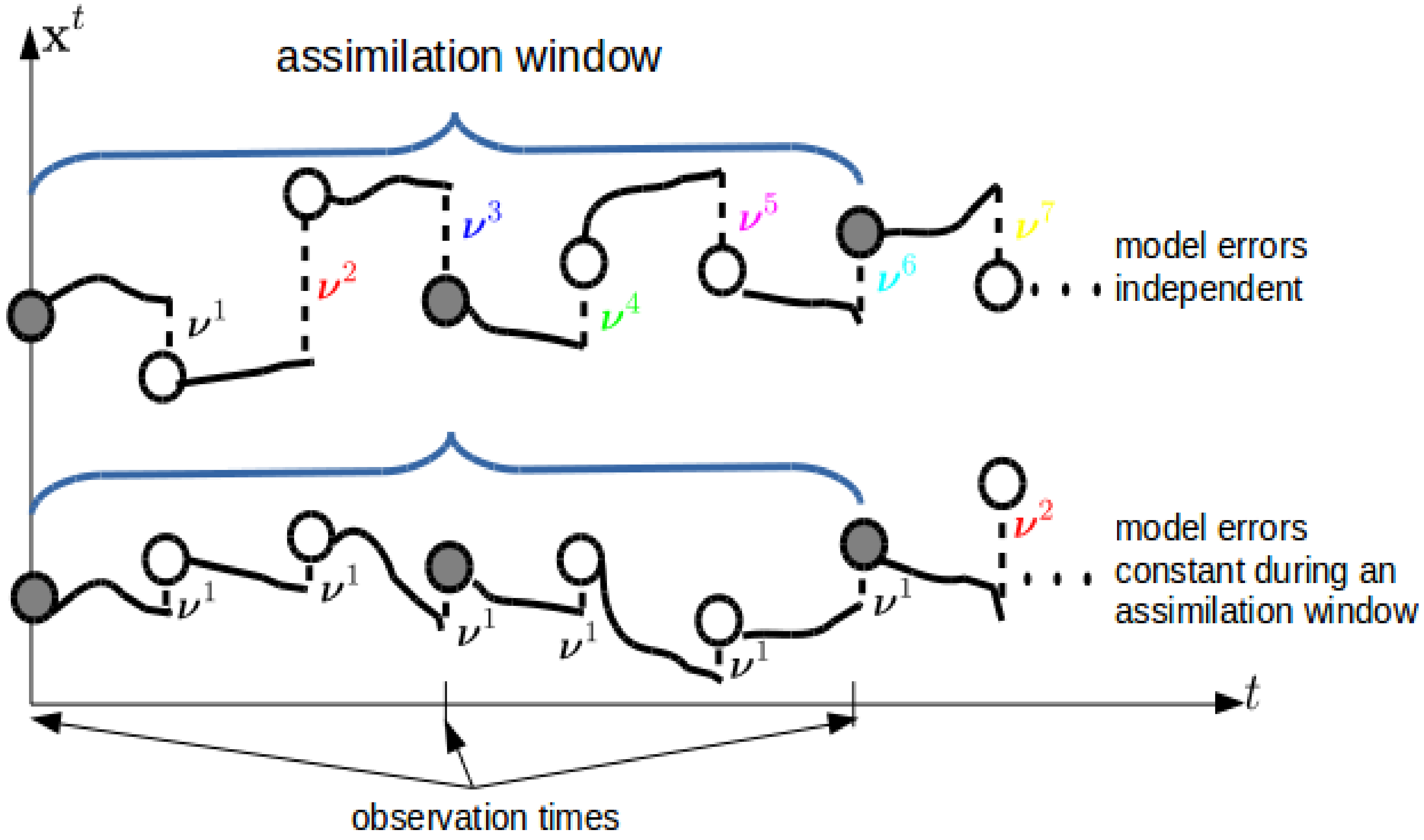
To obtain the **posterior** pdf we can use **Bayes' theorem**.

$$p\left(\mathbf{x}^{0:\tau} | \mathbf{y}^{1,L}\right) = \frac{p\left(\mathbf{y}^{1:L} | \mathbf{x}^{0:\tau}\right) p\left(\mathbf{x}^{0:\tau}\right)}{p\left(\mathbf{y}^{1:L}\right)}$$

2. Model error

$p\left(\mathbf{x}^{0:\tau}\right)$

Consider the two **limiting cases**:



2. Model error

$$\mathbf{x}^t = m^{(t-1)\to t}\left(\mathbf{x}^{t-1}\right) + \mathbf{v}^t$$

In general: $cov(\mathbf{v}^i, \mathbf{v}^j) = \phi\left(|i-j|, \omega\right)\mathbf{Q}$

e.g. $\phi\left(|i-j|, \omega\right) = e^{-\frac{|i-j|}{\omega}}$

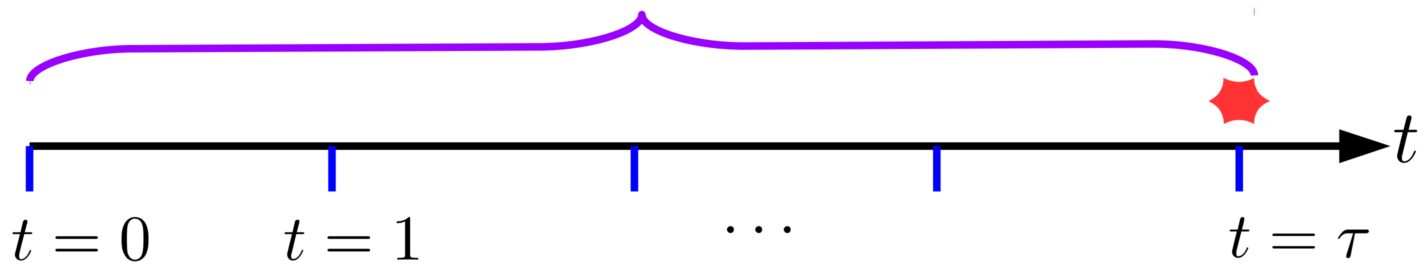
What happens if we use a **wrong time-scale** in DA?

Forecast model vs. **real (imperfect) model**

ω_g ω

More fundamentally, what is the **effect of the memory of the error?**

Simpler problem: Kalman Smoother



Initial conditions: $\mathbf{x}^0 \sim N\left(\mu_x^{0,b}, \mathrm{B}\right)$

Dynamics: $\mathbf{x}^t = \mathrm{M}^{(t-1)\rightarrow t}\mathbf{x}^{t-1} + \mathbf{v}^t \rightarrow \mathbf{v}^t \sim N\left(\mathbf{0}, \mathbf{Q}\right)$

$cov\left(\mathbf{v}^i, \mathbf{v}^j\right) = \phi\left(|i-j|, \omega\right)\mathbf{Q}$

Observation: $\mathbf{y}^l = \mathbf{H}^l\mathbf{x}^t + \boldsymbol{\eta}^l \rightarrow \boldsymbol{\eta}^l \sim N\left(\mathbf{0}, \mathbf{R}\right)$

The WC solution to the KS

Write this as an **extended** problem. $\mathbf{z}^{0:\tau} = \left[(\mathbf{x}^0)^\mathbf{T}, (\mathbf{v}^{1:\tau})^\mathbf{T}\right]^\mathbf{T}$

$$\mathbf{M}^{0:\tau} = \left[\mathbf{M}^{0\to\tau}, \mathbf{M}^{1\to\tau}, \mathbf{M}^{2\to\tau}, \cdots, \mathbf{M}^{(\tau-1)\to\tau}, \mathbf{I}\right]$$

At the time of the **observation**: $\mathbf{x}^\tau = \mathbf{M}^{0:\tau}\mathbf{z}^{0:\tau}$

The **analysis**: $\underline{\mathbf{z}^{0:\tau}|\mathbf{y} \sim N\left(\boldsymbol{\mu}_z^{0:\tau,a}, \mathbf{A}_z^{0:\tau}\right)}$

With **moments**:

$$\boldsymbol{\mu}_z^{0:\tau,a} = \left(\mathbf{I} - \mathbf{K}_z^{0:\tau}\mathbf{H}\mathbf{M}^{0:\tau}\right)\boldsymbol{\mu}_z^{0:\tau,b} + \mathbf{K}_z^{0:\tau}\mathbf{y}$$

$$\mathbf{A}_z^{0:\tau} = \left(\mathbf{I} - \mathbf{K}_z^{0:\tau}\mathbf{H}\mathbf{M}^{0:\tau}\right)\mathbf{D}^{0:\tau}$$

Extended **background/model error covariance**:

$$\mathbf{D}^{0:\tau} = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}^{1:\tau} \end{bmatrix}$$

Where does the time-scale appear?

The **gain**:

$$\mathrm{K}_z^{0:\tau} = \mathrm{D}^{0:\tau}\left(\mathrm{M}^{0:\tau}\right)^{\mathbf{T}}\mathrm{H}^{\mathbf{T}}\left(\Gamma^{\tau}\right)^{-1}$$

Note this is a block-matrix, with elements corresponding to the **different time steps**

The **total covariance** (acts as 'denominator' in the gain):

$$\Gamma^{\tau} = \mathrm{H}\mathrm{B}^{\tau}\mathrm{H}^{\mathbf{T}} + \mathrm{H}\Lambda^{\tau}\mathrm{H}^{\mathbf{T}} + \mathrm{R} \longrightarrow \begin{cases} \mathrm{B}^{\tau} = \mathrm{M}^{0\to\tau}\mathrm{B}\left(\mathrm{M}^{0\to\tau}\right)^{\mathbf{T}} \\ \Lambda^{\tau} = \mathrm{M}^{1:\tau}\mathrm{Q}^{1:\tau}\left(\mathrm{M}^{1:\tau}\right)^{\mathbf{T}} \end{cases}$$

Contribution of **model error** to **total covariance** depends on **memory**:

$$\Lambda^{\tau} = \sum_{i=1}^{\tau}\sum_{j=1}^{\tau}\mathrm{M}^{i\to\tau}\mathrm{Q}(\mathrm{M}^{j\to\tau})^{\mathbf{T}}\phi\left(|i-j|,\omega\right)$$

$$\begin{cases} \lim\limits_{\omega\to 0}\Lambda^{\tau} = \sum_{j=1}^{\tau}\mathrm{M}^{j\to\tau}\mathrm{Q}(\mathrm{M}^{j\to\tau})^{\mathbf{T}}, \\ \lim\limits_{\omega\to\infty}\Lambda^{\tau} = \tilde{\mathrm{M}}^{\tau}\mathrm{Q}\left(\tilde{\mathrm{M}}^{\tau}\right)^{\mathbf{T}} \end{cases}$$

Impacts at different times:

The **gain** for **initial conditions**:

$$\mathbf{K}_x^0 = \mathbf{B}\left(\mathbf{M}^{0\to\tau}\right)^{\mathbf{T}}\mathbf{H}^{\mathbf{T}}\left(\boldsymbol{\Gamma}^{\tau}\right)^{-1}$$

The **gain** for **model-error jumps**:

$$\mathbf{K}_v^j = \mathbf{Q}\left(\sum_{i=1}^{\tau}\left(\mathbf{M}^{i\to\tau}\right)^{\mathbf{T}}\phi(|i-j|,\omega)\right)\mathbf{H}^{\mathbf{T}}\left(\boldsymbol{\Gamma}^{\tau}\right)^{-1}
\left\{\begin{array}{l}
\lim\limits_{\omega\to 0}\mathbf{K}_v^j = \mathbf{Q}\left(\mathbf{M}^{j\to\tau}\right)^{\mathbf{T}}\mathbf{H}^{\mathbf{T}}\left(\boldsymbol{\Gamma}^{\tau}\right)^{-1}, \\[2ex]
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\end{array}\right.$$

The gain for the **state variables at different times**:

$$\mathbf{K}_x^t = \mathbf{M}^{0\to t}\mathbf{K}_x^0 + \sum_{j=1}^{t}\mathbf{M}^{j\to t}\mathbf{K}_v^j$$

This is a **composed** gain.

4. Unidimensional example

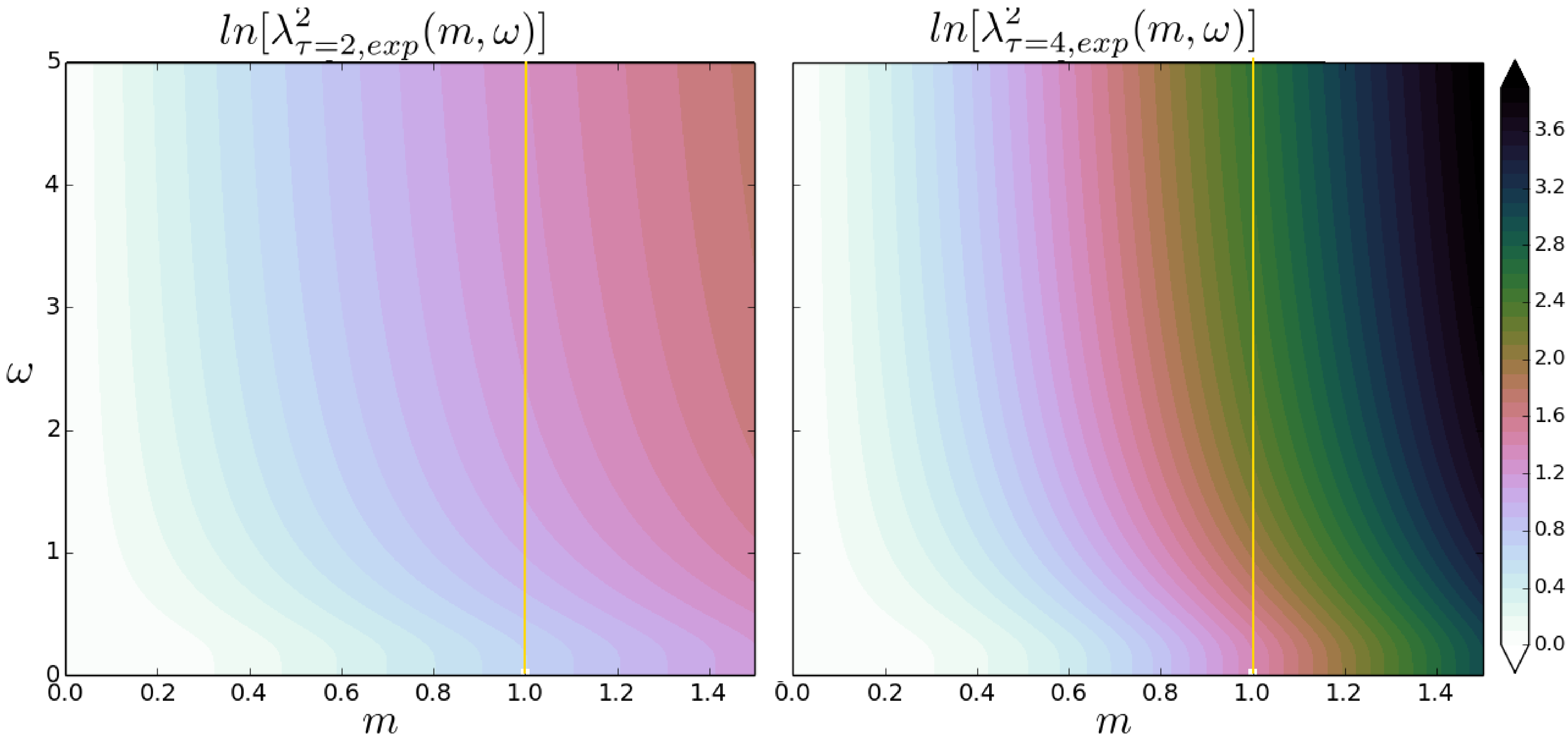
In a simple **uni-dimensional case**, it is easy to visualize the behaviour of

- **model error contribution** to the **total covariance**
- the **gains** at different time steps.

This scalars are plotted as a function of the **model and the time-autocorrelation scale.**

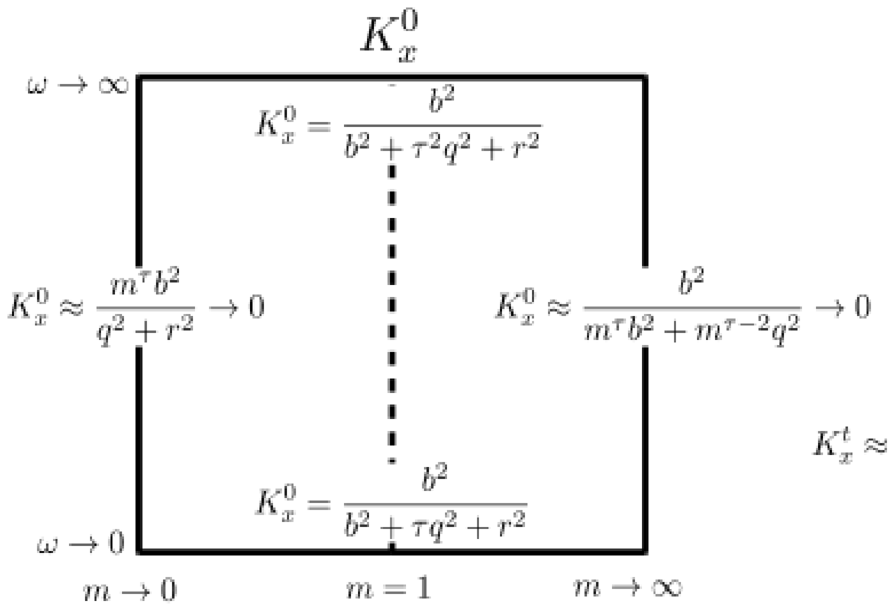
Unidimensional example

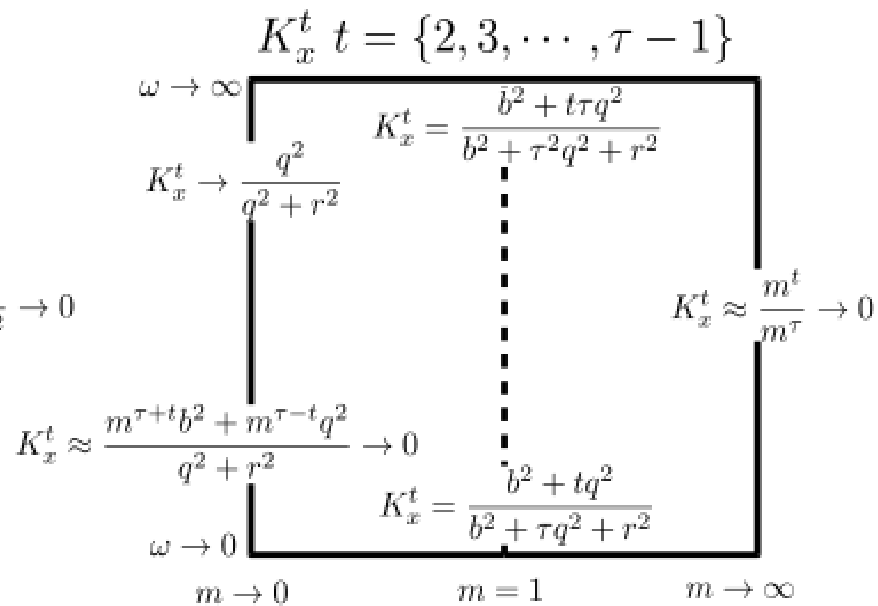
Contribution from model error to the total error covariance.

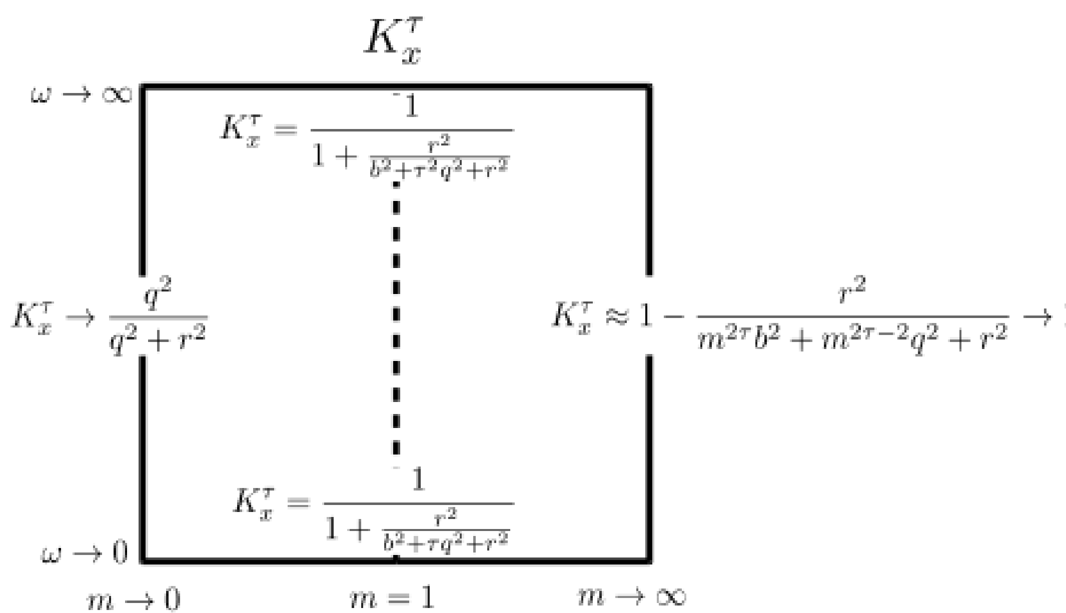


Gains

$$\mathbf{K}_x^t = \mathbf{M}^{0\to t}\mathbf{K}_x^0 + \sum_{j=1}^{t} \mathbf{M}^{j\to t}\mathbf{K}_v^j$$

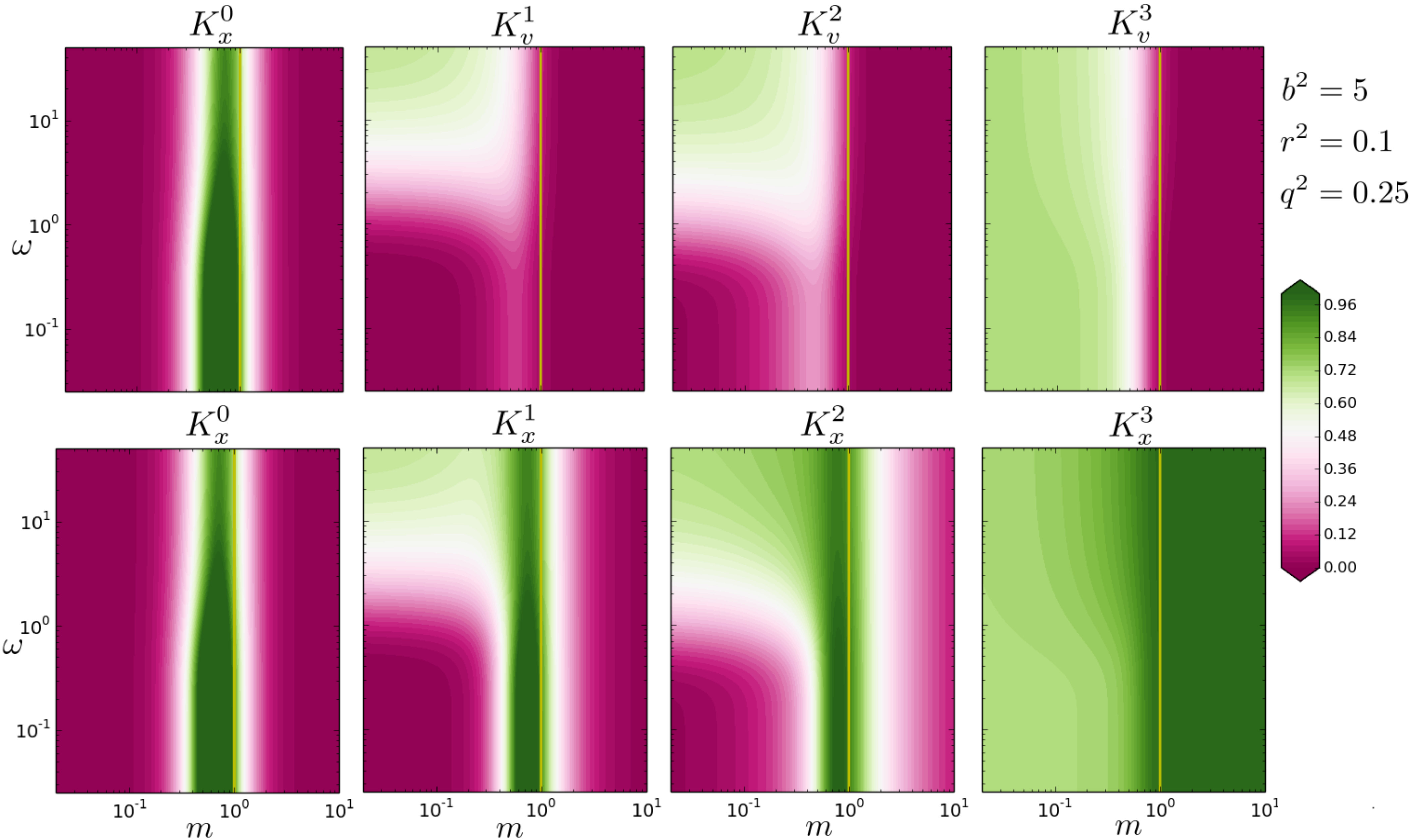






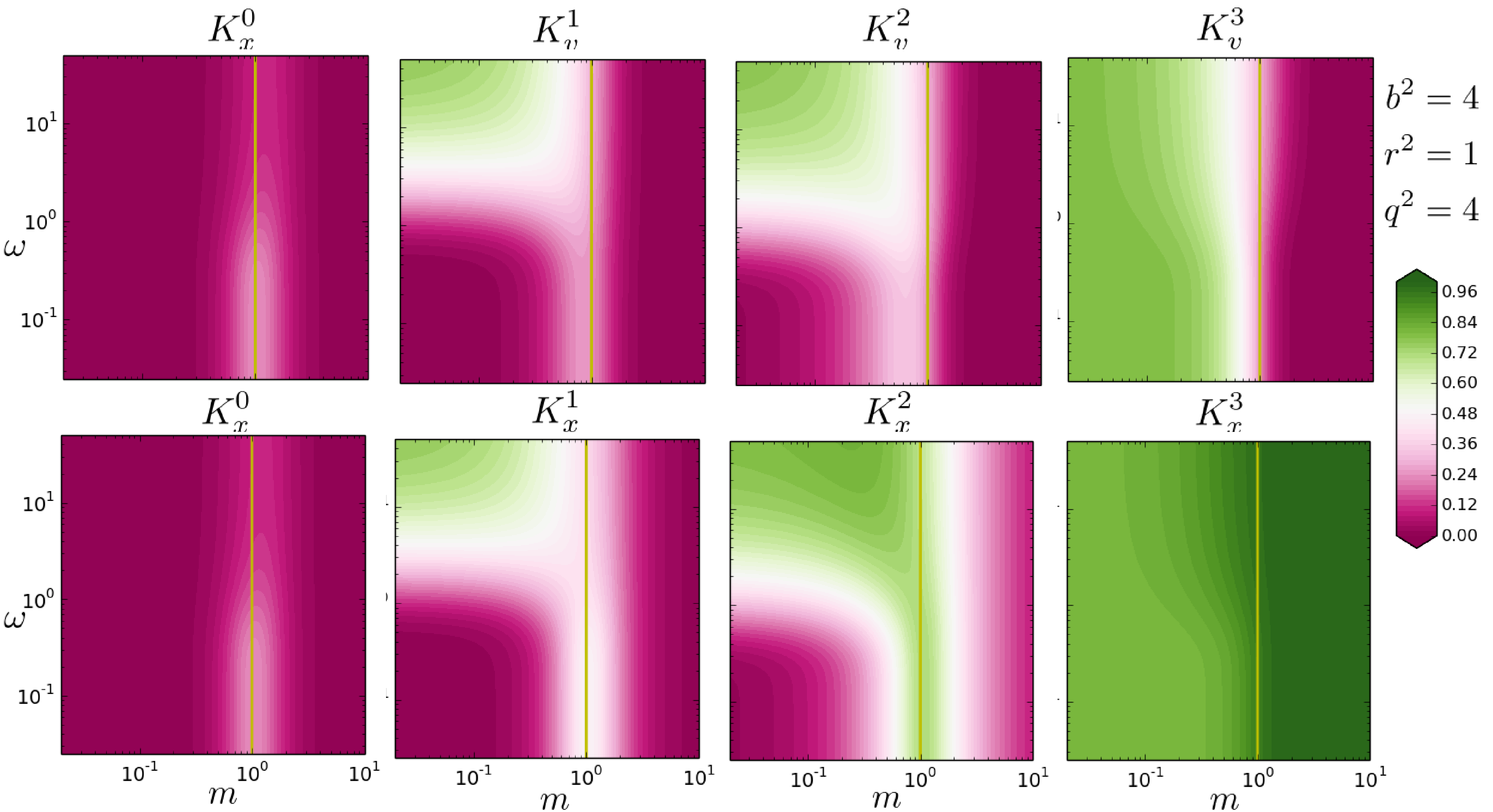
Gains

$$\mathbf{K}_x^t = \mathbf{M}^{0\to t}\mathbf{K}_x^0 + \sum_{j=1}^{t} \mathbf{M}^{j\to t}\mathbf{K}_v^j$$



Gains

$$\mathbf{K}_x^t = \mathrm{M}^{0\to t}\mathbf{K}_x^0 + \sum_{j=1}^{t} \mathrm{M}^{j\to t}\mathbf{K}_v^j$$



5. Introducing errors

The extended background covariance:

$$\mathbf{D}_e^{0:\tau} = \begin{bmatrix} \underset{\uparrow}{\overset{\text{sampling}}{\mathbf{B}_e}} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_g^{1:\tau} \end{bmatrix}$$

Wrong prescription

The total effect of (**indirect**) **sampling errors and a wrong prescription** of the time-scale?

5. Introducing errors

The extended background covariance:

sampling

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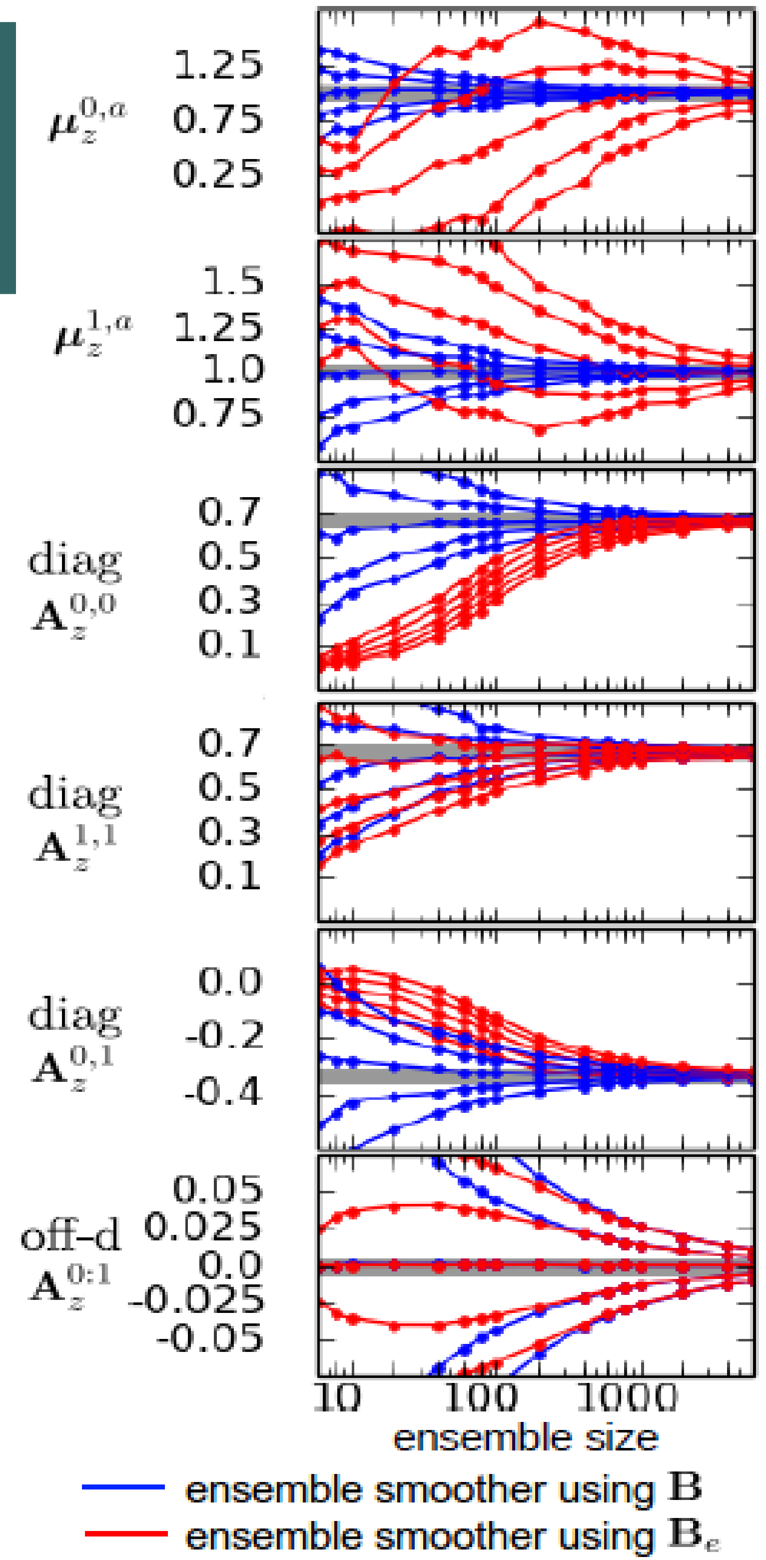


The error-ridden matrices

$$\mathbf{D}_{ge}^{0:\tau}, \mathbf{\Gamma}_{ge}^{\tau}, \mathbf{K}_{ge}^{0:\tau},$$

Once there is an error in the **background error** matrix, this impacts the **total covariance** and the **gain**.

Solution for each analysis member:

$$\underline{\mathbf{z}_{g,n_e}^{0:\tau,a} = \left(\mu_z^{0:\tau,b} + \mathbf{K}_{z,ge}^{0:\tau}\mathbf{d}\right) + \left(\zeta_{g,n_e}^{0:\tau,b} + \mathbf{K}_{z,ge}^{0:\tau}\delta_{g,n_e}\right)}$$

mean perturbation

The exact expressions for the 'ge' expressions are:

$$\mathbf{\Gamma}_{ge}^{\tau} = \mathbf{\Gamma}^{\tau} + \mathbf{\Gamma}_{g\epsilon}^{\tau} = \left(\mathbf{I} + \mathbf{\Gamma}_{g\epsilon}^{\tau}\left(\mathbf{\Gamma}^{\tau}\right)^{-1}\right)\mathbf{\Gamma}^{\tau}$$

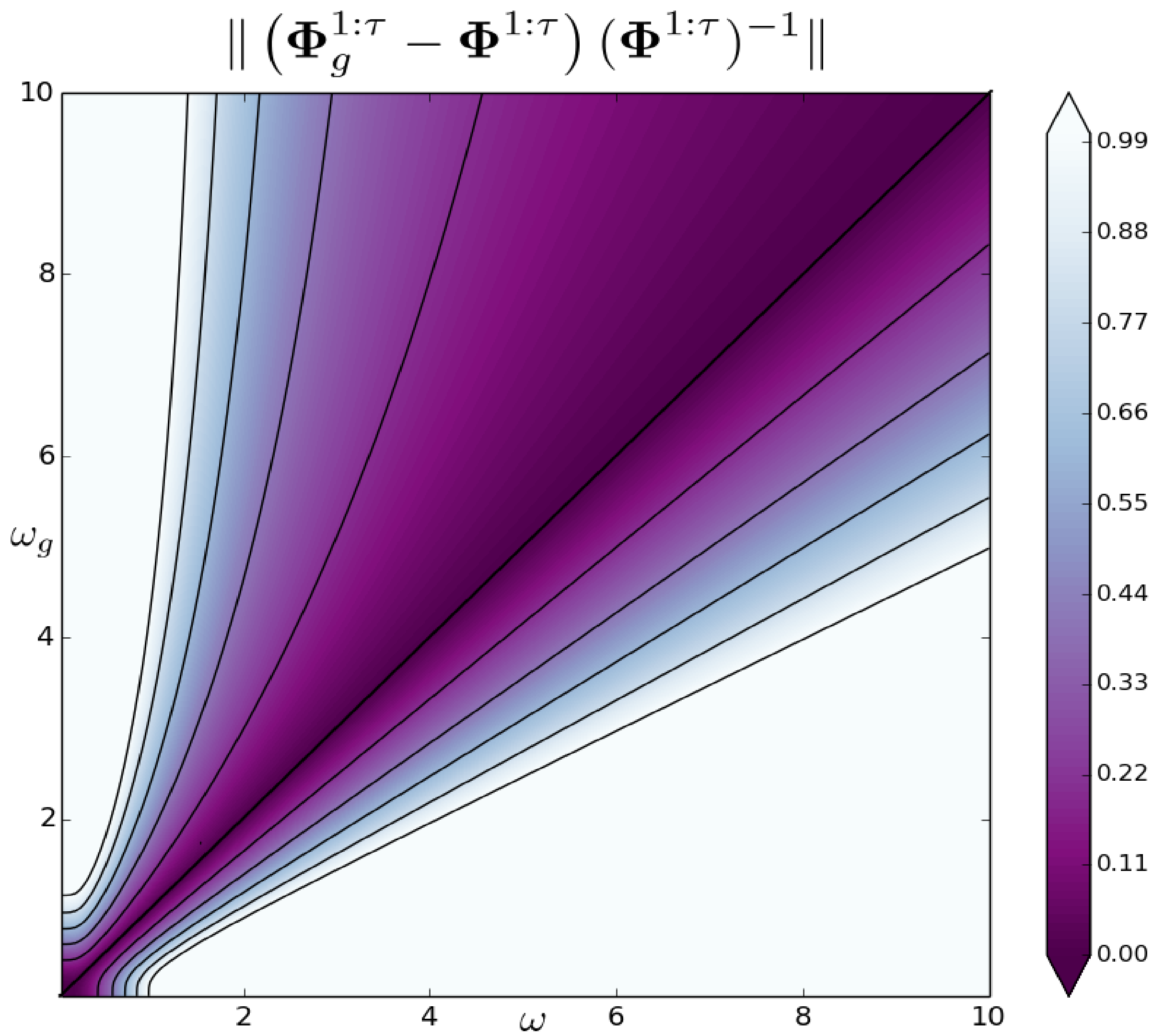
$$\underline{\mathbf{K}_{z,ge}^{0:\tau} = \left(\mathbf{I} + \epsilon_D^{0:\tau}\left(\mathbf{D}^{0:\tau}\right)^{-1}\right)\mathbf{K}_z^{0:\tau}\left(\mathbf{I} + \mathbf{\Gamma}_{g\epsilon}^{\tau}\left(\mathbf{\Gamma}^{\tau}\right)^{-1}\right)^{-1}}$$

6. Small error approximation

$$\|\epsilon_D^{0:\tau}\left(\mathrm{D}^{0:\tau}\right)^{-1}\| \ll 1,$$

Sampling error is 'small'.

Prescription error is 'small'.



How does the **error in the memory affect the norm**?

Small error approximation

The inverse can be linearised.

$$\mathbf{K}^{0:\tau}_{z,ge} \approx \left(\mathbf{I} + \epsilon^{0:\tau}_{D} \left(\mathbf{D}^{0:\tau}\right)^{-1}\right) \mathbf{K}^{0:\tau}_{z} \left(\mathbf{I} - \mathbf{\Gamma}^{\tau}_{g\epsilon} \left(\mathbf{\Gamma}^{\tau}\right)^{-1}\right)$$

And then things become additive:

$$\underline{\mathbf{K}^{0:\tau}_{z,ge} \approx \mathbf{K}^{0:\tau}_{z} + \mathbf{K}^{0:\tau}_{z,g\epsilon}}$$

To first order, the ensemble-based wrong-memory gain is the **sum** of the **correct gain** and an **error part.**

$$\underline{\mathbf{K}^{0:\tau}_{z,g\epsilon} = \epsilon^{0:\tau}_{D} \left(\mathbf{D}^{0:\tau}\right)^{-1} \mathbf{K}^{0:\tau}_{z} - \mathbf{K}^{0:\tau}_{z} \mathbf{\Gamma}^{\tau}_{g\epsilon} \left(\mathbf{\Gamma}^{\tau}\right)^{-1}}$$

Small error approximation

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The components of the **sampling estimators.**

	exact part	direct sampling error	indirect errors error (linear)	indirect sampling error (non-linear)
$\bar{\mathbf{z}}^{0:\tau,a}_{ge}$	$\mu^{0:\tau}_z$	$+\bar{\zeta}^{0:\tau,a}_g$	$+\mathbf{K}^{0:\tau}_{z,g\epsilon}\mathbf{d}$	$+\mathbf{K}^{0:\tau}_{z,g\epsilon}\bar{\delta}_g$
$\mathbf{A}^{0:\tau}_{z,ge}$	$\mathbf{A}^{0:\tau}_z$	$+\left(\mathbf{I} - \mathbf{K}^{0:\tau}_z \mathbf{H}\mathbf{M}^{0:\tau}\right)\epsilon^{0:\tau}_D$	$-\mathbf{K}^{0:\tau}_{z,g\epsilon}\mathbf{H}\mathbf{M}^{0:\tau}\mathbf{D}^{0:\tau}$	$-\left(\mathbf{H}\mathbf{M}^{0:\tau}\epsilon^{0:\tau}_D\right)^{\mathbf{T}}\left(\mathbf{\Gamma}^{\tau}\right)^{-1}\mathbf{H}\mathbf{M}^{0:\tau}\epsilon^{0:\tau}_D$

7. Conclusions, future work

- We have explored the effect of time **auto-correlated model error** in the **Kalman Smoother**.

- The impact of the observations over a window depend strongly upon the magnitude of the model. For **shrinking models**, the **magnitude of the 'ememory' matters considerably**.

- We have explored the errors created by: **sampling estimators of the background error covariance**, and the **error in prescribing the model error covariance**. Their interactions are not trivial, and they affect both mean and covariance.

- This can be useful when one has to use **fixed (or slow-evolving) model errors for computational reasons**.

- What happens with **cycling** and with **non-linear models**? (current MSc/PhD Student)

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